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Some Remarks on the Groetsch–Shisha Theorem

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1. INTRODUCTION

Let $f \in C[0, 1]$, $\|f\| := \max_{0 \leqslant x \leqslant 1} |f(x)|$. For $f \in C[0, 1]$, let $B_n(f, x)$ be the Bernstein polynomial of order n. Groetsch and Shisha [1] proved that if $f \in C[0, 1]$ has a bounded derivative on $(0, 1)$, then

$$|B_n(f, x) - f(x)| \leqslant x\omega(f', 1/n), \tag{1.1}$$

where $\omega(g, \cdot)$ is the usual modulus of continuity of g.

We investigate here estimations of Groetsch–Shisha type for more general positive linear operators.

2. MAIN RESULTS

THEOREM 1. *For a sequence of linear operators L_n: $C[0, 1] \to C[0, 1]$ the estimate*

$$|L_n(f, x) - f(x)| \leqslant M_1 x\omega(f', \lambda_n) \tag{2.1}$$

for some $\lambda_n \in (0, 1]$ holds for all $f \in C^1[0, 1]$ with M_1 independent of f, n, and x if and only if

(i) *for all $f \in C^1[0, 1]$,*

$$|L_n(f, x) - f(x)| \leqslant M_2 x \|f'\|, \tag{2.2}$$

(ii) *and for all $f \in C^2[0, 1]$,*

$$|L_n(f, x) - f(x)| \leqslant M_3 x\lambda_n \|f''\|. \tag{2.3}$$

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Proof. (⇒). From (2.1) it follows that for $f \in C^1[0, 1]$

$$|L_n(f, x) - f(x)| \leqslant 2M_1 x \, \|f'\|.$$

If $f \in C^2[0, 1]$, then from (2.1) and $\omega(f', \lambda_n) \leqslant \lambda_n \, \|f''\|$ we have

$$|L_n(f, x) - f(x)| \leqslant M_1 x \lambda_n \, \|f''\|.$$

(⇐). For $f \in C^1[0, 1]$ the K-functional

$$K(f', \delta) = \inf_{\phi \in C^1[0,1]} (\|f' - \phi\| + \delta \, \|\phi'\|) \tag{2.4}$$

is equivalent to $\omega(f', \delta)$. We choose ϕ so that

$$\|f' - \phi\| + \delta \, \|\phi'\| \leqslant 2K(f', \delta) \leqslant M_4 \omega(f', \delta), \tag{2.5}$$

and define $\psi(t) = \int_0^t \phi(s) \, ds$. As L_n is linear, we now estimate $L_n(f - \psi, x)$ and $L_n(\psi, x)$ using (i) and (ii), respectively, and add the results together to complete the proof. ∎

THEOREM 2. *Assume that a sequence of positive linear operators* L_n: $C[0, 1] \to C[0, 1]$ *satisfies the following for some* $\lambda_n \in (0, 1]$.

(i) *For each* $f \in C[0, 1]$,

$$L_n(f, 0) = f(0). \tag{2.6}$$

(ii) *For* $f \in C^1[0, 1]$

$$\left\| \frac{d}{dx} L_n(f, x) \right\| \leqslant M_5 \, \|f'\|. \tag{2.7}$$

(iii)

$$L_n((t - x)^2, x) \leqslant M_6 x \lambda_n. \tag{2.8}$$

Then for each $f \in C^1[0, 1]$,

$$|L_n(f, x) - f(x)| \leqslant M_7 \{x \omega(f', \lambda_n) + |L_n(t, x) - x| \cdot \|f'\|\}. \tag{2.9}$$

Proof. Taking $f(x) = c$ in (2.7) we obtain $\|(d/dx) L_n(c, x)\| = 0$. Thus $L_n(c, x)$ is a constant and from (2.6), $L_n(c, x) = c$. If f is linear then (2.9) obviously holds as long as $M_7 \geqslant 1$. Assume $f \in C^1[0, 1]$ and f' is not a constant. Thus $\omega(f', \lambda_n) > 0$ (since $\lambda_n > 0$). From (2.4) and (2.5) we have

$$2K(f', \lambda_n) \leqslant M_4 \omega(f', \lambda_n).$$

Hence there exists a $\phi \in C^1[0, 1]$ such that

$$\|f' - \phi\| + \lambda_n \|\phi'\| < M_4 \omega(f', \lambda_n). \tag{2.10}$$

Thus

$$\|f' - \phi\| < 2M_4 \|f'\|, \qquad \|\phi\| < (2M_4 + 1) \|f'\|. \tag{2.11}$$

Define $\psi(t) = \int_0^t \phi(s)\, ds$. From (2.6) we have

$$L_n(f - \psi, 0) = f(0) - \psi(0).$$

Therefore,

$$\begin{aligned} |L_n(f, x) - f(x)| \leqslant{} & |L_n(f - \psi, x) - L_n(f - \psi, 0) \\ & - [(f(x) - \psi(x)) - (f(0) - \psi(0))]| \\ & + |L_n(\psi, x) - \psi(x) - \psi'(x)(L_n(t, x) - x)| \\ & + \|\psi'\| \cdot |L_n(t, x) - x| \\ =:{} & I_1(x) + I_2(x) + I_3(x). \end{aligned}$$

From (2.7),

$$\begin{aligned} I_1(x) &\leqslant \left| \int_0^x \left(\frac{d}{ds} L_n(f - \psi, s) \right) ds \right| + \left| \int_0^x (f'(s) - \psi'(s))\, ds \right| \\ &\leqslant x \left\| \frac{d}{ds} L_n(f - \psi, s) \right\| + x \|f' - \psi'\| \\ &\leqslant (M_5 + 1)\, x \|f' - \phi\|. \end{aligned} \tag{2.12}$$

From (2.8) and since L_n is a positive linear operator,

$$I_2(x) \leqslant \tfrac{1}{2} L_n((t - x)^2, x) \cdot \|\psi''\| \leqslant \tfrac{1}{2} M_6 x \lambda_n \|\phi'\|. \tag{2.13}$$

From (2.11),

$$I_3(x) < (2M_4 + 1)\, |L_n(t, x) - x| \cdot \|f'\|. \tag{2.14}$$

Combining (2.12), (2.13), and (2.14), and using (2.10) proves the theorem. ∎

EXAMPLE. Let $n \in \mathbb{N}$ and $\alpha(n) \geqslant 0$. For $f \in C[0, 1]$ the Sikkema–Bernstein polynomial [2] is defined by

$$D_n(f, x) = \sum_{i=0}^{n} f\left(\frac{i}{n + \alpha(n)} \right) p_{n,i}(x), \qquad p_{n,i}(x) = \binom{n}{i} x^i (1 - x)^{n-i}.$$

From the above we obtain:

THEOREM 3. *If $Q \subset \mathbb{N}$ and for $n \in Q$ we have $0 < 1/n + \alpha^2(n)/n^2 \leqslant 1$, then for each $n \in Q$, $x \in [0, 1]$ and $f \in C^1[0, 1]$ we have*

$$|D_n(f, x) - f(x)| \leqslant M_8 x \left\{ \omega\left(f', \frac{1}{n} + \frac{\alpha^2(n)}{n^2}\right) + \frac{\alpha(n)}{n} \|f'\| \right\}.$$

If $\alpha(n) = 0$ for all n, then we obtain

$$|B_n(f, x) - f(x)| \leqslant M_8 x \omega(f', 1/n).$$

REFERENCES

1. C. W. GROETSCH AND O. SHISHA, On the degree of approximation by Bernstein polynomials, *J. Approx. Theory* **14** (1975), 317–318.
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